Applications of Lipschitz Methods in Real Algebraic Geometry

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Contents

1 Introduction

This project is a study of Lipschitz continuous functions in real algebraic geometry. These maps are characterised by their bounded rate of change. This property not only provides a natural framework for studying the metric behaviour of functions and sets, but also plays a crucial role in understanding the structure of singularities, knots, and semialgebraic varieties.

In Chapter 2, we establish foundations by investigating the theory of Lipschitz and Hölder continuous maps. We define these maps and explore their key properties such as boundedness, their behaviour under composition and addition, as well as the concept of bi-Lipschitz equivalence. This chapter lays the groundwork for our later investigations.

In Chapter 3 we introduce fundamental concepts from knot theory. We specifically focus on their role in real algebraic geometry. We define knots and links, discuss knot diagrams, and also detail Reidemeister moves, which generate ambient isotopies. These topics provide a base for the study of knots from a Lipschitz perspective.

Chapter 4 looks at semialgebraic sets and semialgebraic maps. After defining these sets and presenting examples, we examine some important properties. We investigate the concept of Lipschitz normal embeddings which is essential for understanding the geometry of semialgebraic sets.

Chapter 5 is dedicated to the study of β-Hölder triangles and β-horns. These serve as models for the behaviour of real algebraic varieties near singular points. We introduce the standard β-Hölder triangle and prove that it is normally embedded. This analysis is then extended to the more complex geometry of β-horns, which play a key role in describing local Lipschitz geometry in higher dimensions. We also look at the E_8 singularity, an example of an isolated singularity in real algebraic geometry. We introduce and study the concept of distortion for embedded curves and knots. We derive lower bounds for the distortion of closed curves and analyse the implications of these bounds.

Chapter 6 looks at the role of tangent cones in Lipschitz geometry. We prove that under the hypothesis of Lipschitz normal embedding, the tangent cone of a set inherits this property.

Finally, Chapter 7 states and proves the results of Universality Theorem, which links Lipshchitz geometry with surface singularities and knots.

Through these chapters, we demonstrate how Lipschitz methods provide a framework to tackle problems in real algebraic geometry and in part, they can be used in the study of singularities through the analysis of knots and conical structures.

2 Lipschitz Analysis

2.1 Lipschitz Maps

Let $f : X \to Y$ be a function between two metric spaces (X, d_X) and (Y, d_Y)

Definition 2.1. Let $\alpha \in \mathbb{R}_{\geq 0}$ be a positive real number. The mapping $f : X \to Y$ is said to be *α-Hölder continuous* if there exists a $L \geq 0$ such that

$$d_Y(f(a), f(b)) \leq L(d_X(a, b))^\alpha$$

for all $a, b \in X$.

When $\alpha = 1$ we have the natural condition that is called Lipschitz continuity.

Definition 2.2. $f : X \to Y$ is Lipschitz continuous if there exists a $K \geq 0$, known as a *Lipschitz constant* of f, such that

$$d_Y(f(a), f(b)) \leq K d_X(a, b)$$

for all $a, b \in X$.

Definition 2.3. A *K-Lipschitz* function is a Lipschitz function f with Lipschitz constant K.

Definition 2.4. The *Lipschitz norm* is the minimum of the Lipschitz constants. We denote this smallest possible Lipschitz constant of f by:

$$\mathrm{LIP}(f) = \inf\{K : d(f(a), f(b)) \leq K d(a, b) \text{ for all } a, b \in X\}.$$

Alternatively, we can say the Lipschitz norm $\|f\|_{\mathrm{Lip}}$ of $f : X \to Y$ is the supremum of the absolute difference quotients

$$\frac{d(f(a), f(b))}{d(a, b)}$$

for $a \neq b$ in X.

Definition 2.5. The map $f : X \to Y$ is *short* if its Lipschitz norm is at most 1. It is a *contraction mapping* if its Lipschitz norm is strictly less than 1.

2.2 Lipschitz Properties

Theorem 2.6. *Lipschitz functions are uniformly continuous.*

Proof. Let $f : X \to Y$ be a map between two metric spaces (X, d_X) and (Y, d_Y) with Lipschitz constant $K > 0$. Let $\epsilon > 0$. If $x_1, x_2 \in X$ with $d(x_1, x_2) < \frac{\epsilon}{K}$ then we have,

$$d_Y(f(x_1), f(x_2)) \leq K d(x_1, x_2) < \epsilon$$

Thus, f is uniformly continuous. □

Remark 2.7. The converse is not true. Not all uniformly continuous functions are Lipschitz. Consider the function $\sqrt{\cdot} : [0,1] \to [0,1]$. This function is continuous as it is the inverse of the continuous function $x \mapsto x^2$ defined on the interval $[0,1]$. $\sqrt{\cdot}$ is uniformly continuous since $[0,1]$ is a closed and bounded and hence compact set.
However, if $x > 0$, then

$$\frac{|\sqrt{x} - \sqrt{0}|}{|x - 0|} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}.$$

As $x \to 0^+$, this quotient is unbounded, and therefore $\sqrt{\cdot}$ is not Lipschitz.

Theorem 2.8. *The composition of Lipschitz maps are Lipschitz.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be Lipschitz functions. Let K_f and K_g be the Lipschitz constants of f and g respectively. Thus, $|f(x) - f(y)| \leq K_f|x - y|$ and $|g(x) - g(y)| \leq K_g|x - y|$ for any $x, y \in \mathbb{R}$. Then,

$$|(f \circ g)(x) - (f \circ g)(y)| = |f(g(x)) - f(g(y))|$$

$$\leq K_f|g(x) - g(y)|$$
$$\leq K_f K_g|x - y|$$

so $f \circ g$ is Lipschitz with constant $K = K_f K_g$. □

Theorem 2.9. *The sum of Lipschitz maps are Lipschitz.*

Proof. With K_f and K_g as above,

$$|(f + g)(x) - (f + g)(y)| = |f(x) + g(x) - f(y) - g(y)|$$

$$= |f(x) - f(y) + g(x) - g(y)|$$
$$\leq |f(x) - f(y)| + |g(x) - g(y)|$$
$$\leq K_f|x - y| + K_g|x - y|$$
$$= (K_f + K_g)|x - y|$$

so $f + g$ is Lipschitz with constant $K = K_f + K_g$. □

2.3 Boundedness

We recall the definition of a bounded function.

Definition 2.10. A function $f : X \to \mathbb{R}$ is *bounded* if there is a constant $M > 0$ such that $|f(x)| \leq M$ for all $x \in M$.

Theorem 2.11. *Any Lipschitz function $f : [a, b] \to \mathbb{R}$ defined on an interval of the form $[a, b]$ is a bounded function.*

Proof. For any $x \in [a, b]$,

$$|f(x) - f(a)| \leq K|x - a| \leq K|b - a|$$

and

$$|f(x)| = |f(x) - f(a) + f(a)| \leq |f(x) - f(a)| + |f(a)| \leq K|b - a| + |f(a)|$$

So, take $M = K|b - a| + |f(a)|$, and thus f is bounded. □

Theorem 2.12. *An everywhere differentiable function $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, with $K = \sup |f'(x)|$ if and only if it has a bounded first derivative.*

Proof. Assume that f is a K-Lipschitz function. So $|f(x + h) - f(x)| \leq K|h|$, $\forall x, h \in \mathbb{R}$, which is equivalent to $\left|\frac{f(x+h)-f(x)}{h}\right| < K$. By taking the limit, $|f'(x)| \leq K$.
For the converse, we use the Mean Value Theorem. Let $x, y \in \mathbb{R}$, there exists $c \in [x, y]$ such that $f(x) - f(y) = (x - y)f'(c)$ and now use the fact that $|f'(c)| \leq K$. □

Example 2.13. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sqrt{x^2 + 3}$. Thus, $f'(x) = \frac{x}{\sqrt{x^2+3}}$ and we have $\sup |f'(x)| = 1$. We set $K = 1$ and see that $f(x)$ is Lipschitz. Similarly, the continuously differentiable sine function $g(x) = \sin(x)$ has derivative $g'(x) = \cos(x)$. Thus $K = \sup |g'(x)| = \sup |\cos(x)| = 1$ and $g(x)$ is Lipschitz continuous.

Remark 2.14. $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is not Lipschitz. Fix $x = 0$, and consider a sequence tending to ∞, say $\{n\}$. Then

$$\frac{|f(n) - f(0)|}{|n - 0|} = \frac{n^2}{n} = n$$

which cannot be bounded by any fixed Lipschitz constant K.

2.4 Bi-Lipschitz Functions

Definition 2.15. Let $f : X \to Y$ be a function between two metric spaces (X, d_X) and (Y, d_Y). f is said to be *bi-Lipschitz* if f is Lipschitz, injective and its inverse is Lipschitz. When f is bi-Lipschitz and surjective, we say that X and Y are *bi-Lipschitz equivalent.*

Theorem 2.16. *A function $f : X \to Y$ is bi-Lipschitz if and only if there exists some $K > 0$ such that*

$$\frac{1}{K} d_1(x_1, x_2) \leq d_2\left(f(x_1), f(x_2)\right) \leq K d_1(x_1, x_2)$$

for all $x_1, x_2 \in X$.

Proof. Suppose that $f : X \to Y$ is bi-Lipschitz. Thus, f is Lipschitz with constant K_1 and f^{-1} is Lipschitz with constant K_2. Let $K = \max\{K_1, K_2\}$. Then, for any $x_1, x_2 \in X$, we have

$$d_2(f(x_1), f(x_2)) \leq K_1 d_1(x_1, x_2) \leq K d_1(x_1, x_2)$$

Similarly,

$$d_1(x_1, x_2) = d_1\left(f^{-1}(f(x_1)), f^{-1}(f(x_2))\right) \leq K_2\, d_2(f(x_1), f(x_2)) \leq K\, d_2(f(x_1), f(x_2))$$

Dividing the latter by K and combining, we see that

$$\frac{1}{K} d_1(x_1, x_2) \leq d_2(f(x_1), f(x_2)) \leq K d_1(x_1, x_2)$$

□

2.5 Extension of Lipschitz Mappings

Theorem 2.17. *Assume $A \subset \mathbb{R}^n$, and let $f : A \to \mathbb{R}^m$ be Lipschitz continuous with Lipschitz constant K. Then there exists a Lipschitz continuous function $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^m$ with Lipschitz constant $\tilde{K}$ such that*

1. *$\tilde{f} = f$ on A,*
2. *$\tilde{K} \leq \sqrt{m}\, K$.*

Proof. 1. Assume $f : A \to \mathbb{R}$ and then define $\overline{f}(x) := \inf_{a \in A}\{f(a) + K|x - a|\}$ for $x \in \mathbb{R}^n$. If $b \in A$, then we have that $\overline{f}(b) = f(b)$. We see that for all $a \in A$, $f(a) + K|b - a| \geq f(b)$, while clearly $\overline{f}(b) \leq f(b)$.
For any $x, y \in \mathbb{R}^n$, we have that $\overline{f}(x) \leq \inf_{a \in A}\{f(a) + K(|y-a| + |x-y|)\} = \overline{f}(y) + K|x-y|$. Similarly, $\overline{f}(y) \leq \overline{f}(x) + K|x - y|$.
2. In the general case where $f : A \to \mathbb{R}^m$, $f = (f^1, \ldots, f^m)$, we define $\overline{f} := (\overline{f^1}, \ldots, \overline{f^m})$. Then we have $|\overline{f}(x) - \overline{f}(y)|^2 = \sum_{i=1}^m |\overline{f^i}(x) - \overline{f^i}(y)|^2 \leq m(K)^2|x - y|^2$. □

2.6 Arzelà–Ascoli Theorem

We recall the definitions of a uniformly bounded sequence of functions and a uniformly equicontinuous sequence of functions before we examine the Arzelà–Ascoli Theorem.

Definition 2.18. A sequence $\{f_n\}_{n\in\mathbb{N}}$ of continuous functions on an interval $I = [a, b]$ is *uniformly bounded* if there exists a number M such that

$$|f_n(x)| \leq M$$

for every function $f_n \in \{f_n\}_{n\in\mathbb{N}}$ and every $x \in [a, b]$.

Definition 2.19. The sequence is said to be *uniformly equicontinuous* if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f_n(x) - f_n(y)| < \varepsilon$$

whenever $|x - y| < \delta$ for all functions f_n in the sequence.

We first cover the the general theorem before generalising for Lipschitz functions.

Theorem 2.20. *(Arzelà-Ascoli). Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued continuous functions defined on a closed and bounded interval [a, b] of the real line. If this sequence is uniformly bounded and uniformly equicontinuous, then there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ that converges uniformly.*

Proof. Let $I = [a, b] \subset \mathbb{R}$ be a closed and bounded interval. If $\mathcal{F}$ is an infinite set of functions $f : I \to \mathbb{R}$ which is uniformly bounded and equicontinuous, then there is a sequence f_n of elements of $\mathcal{F}$ such that f_n converges uniformly on I.
Let $\{f_n\}$ be any sequence in $\mathcal{F}$. We will show that there exists a uniformly convergent subsequence. First, fix a countable set $\{x_i\}_{i\in\mathbb{N}}$. Now $\mathcal{F}$ is uniformly bounded so the set of points $\{f(x_1)\}_{f\in\mathcal{F}}$ is bounded. Thus by Bolzano-Weierstrass theorem, there exists a subsequence $\{f_{n_1}\}$ of functions in $\mathcal{F}$ such that $\{f_{n_1}(x_1)\}$ converges. Similarly, for $\{f_{n_1}(x_2)\}$ there exists a subsequence $\{f_{n_2}\}$ of $\{f_{n_1}\}$ such that $\{f_{n_2}(x_2)\}$ converges. Continuing inductively, we obtain a nested sequence of subsequences such that for every $k \geq 1$, $\{f_{n_k}\}$ converges at $x_1, x_2 ... x_k$. We then form a diagonal subsequence $\{f_m\}$ where the mth term f_m is the mth term in the mth subsequence $\{f_{n_m}\}$. By construction, f_m converges at every rational point of I. Therefore, for $\varepsilon > 0$ and rational x_k in I, there exists an integer $N = N(\varepsilon, x_k)$ such that

$$|f_n(x_k) - f_m(x_k)| < \frac{\varepsilon}{3}, \quad n, m \geq N$$

$\mathcal{F}$ is equicontinuous, therefore, for this fixed ε and for every x in I, there exists an open interval U_x containing x such that

$$|f(s) - f(t)| < \frac{\varepsilon}{3}$$

for all $f \in \mathcal{F}$ and all s, t in I such that $s, t \in U_x$. The collection of intervals U_x, $x \in I$, forms an open cover of I. Using the Heine–Borel theorem on the closed and bounded I, we can see that I is compact. This means that this covering admits a finite subcover $U_1, \ldots, U_J$. Now there exists a natural number K such that each open interval U_j, $1 \leq j \leq J$, contains a rational x_k with $1 \leq k \leq K$. Finally, for any $t \in I$, there exist j and k so that t and x_k belong to the same interval U_j. For this k,

$$\begin{aligned} |f_n(t) - f_m(t)| &\leq |f_n(t) - f_n(x_k)| + |f_n(x_k) - f_m(x_k)| + |f_m(x_k) - f_m(t)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

for all $n, m > N = \max\{N(\varepsilon, x_1), \ldots, N(\varepsilon, x_K)\}$. Therefore, the sequence $\{f_n\}$ is uniformly Cauchy and thus converges uniformly to a continuous function, as claimed. □

We can now generalise Arzelà-Ascoli Theorem to a family of Lipschitz functions:

Corollary 2.21. *If $\{f_n\}$ is a uniformly bounded sequence of real-valued functions on $[a, b]$ such that each f_n is Lipschitz continuous with the same Lipschitz constant K:*

$$|f_n(x) - f_n(y)| \leq K|x - y|$$

for all $x, y \in [a, b]$ and all f_n, then there exists a subsequence that converges uniformly on $[a, b]$.

Theorem 2.22. *Let $\{f_n\}$ be a sequence of uniformly continuous functions from (X, d_X) into (Y, d_Y) If each f_n is a Lipschitz function with constants K_n and $\sup_n K_n < \infty$, then f is a Lipschitz function.*

Proof. Suppose the f_n's are Lipschitz functions with constant K_n. Then we have $d_Y(f_n(x), f_n(y)) \leq K_n d_X(x, y)$ for all $x, y \in X$. Let $K = \sup_n K_n$. Pick N large enough such that $d_Y(f(x), f_N(x)) < \varepsilon/2$ for all $x \in X$. Then, we have

$$\begin{aligned} d_Y(f(x), f(y)) &\leq d_Y(f(x), f_N(x)) + d_Y(f_N(x), f_N(y)) + d_Y(f_N(y), f(y)) \\ &< \varepsilon/2 + K_N d_X(x, y) + \varepsilon/2 \\ &\leq \varepsilon + K d_X(x, y) \end{aligned}$$

So we have that $d_Y(f(x), f(y)) \leq K d_X(x, y)$. Thus, f is a Lipschitz function. □

Example 2.23. We can illustrate the necessity of the uniform equicontinuity condition in the Arzelà-Ascoli theorem with the following example. Consider the family $\mathcal{F} = \{f_n\}_{n=1}^{\infty}$ of functions $f_n : [0, 1] \to \mathbb{R}$ defined by

$$f_n(x) = \sin(nx)$$

Each f_n is Lipschitz with Lipschitz constant $K_n = n$. The Lipschitz constants are unbounded. Since $|\sin(nx)| \leq 1$, the family $\mathcal{F}$ is uniformly bounded. The family $\mathcal{F}$ is *not* equicontinuous on $[0, 1]$. For any $\delta > 0$, choose $x = 0$ and $y = \delta$. Then

$$|f_n(x) - f_n(y)| = |\sin(0) - \sin(n\delta)| = |\sin(n\delta)| \leq n\delta$$

As $n \to \infty$, $n\delta \to \infty$ unless $\delta = 0$. Therefore, for any fixed $\varepsilon > 0$, we cannot find a $\delta > 0$ such that $|f_n(x) - f_n(y)| < \varepsilon$ for all n.

2.7 Differentiability of Lipschitz Functions

We recall the definition of differentiability:

Definition 2.24. The function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *differentiable* at $x \in \mathbb{R}^n$ if there exists a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{y \to x} \frac{|f(y) - f(x) - L(y - x)|}{|y - x|} = 0$$

Remark 2.25. Lipschitz continuous functions do not have to be everywhere differentiable. For example, consider $f(x) = |x|$ on the reals. This function is not differentiable at zero because

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1 \neq -1 = \lim_{x \to 0^-} \frac{-x}{x} = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x}$$

However, f is Lipschitz with constant 1 which we prove as follows. Without loss of generality, we can assume $x \leq y$ since $||x| - |y|| = ||y| - |x|| \quad \text{and} \quad |x - y| = |y - x|$, which means we are free to exchange x and y. We thus have three cases.

1. If $0 \leq x \leq y$, then $||y| - |x|| = y - x = |y - x|$.
2. If $x < 0 \leq y$, then $||y| - |x|| = y + x < y - x = |y - x|$.
3. If $x \leq y < 0$, then $||y| - |x|| = y - x = |y - x|$.

This establishes $||y| - |x|| \leq |y - x|$ in all cases, and so the function is Lipschitz.

Theorem 2.26. *(Rademacher's Theorem). Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$ Lipschitz continuous. Then f is differentiable at almost every $x \in U$.*

We outline the proof from [11]

Proof. We begin by assuming without loss of generality that $m = 1$. Since differentiability is a local property, we can further assume that f is Lipschitz continuous. Fix any unit vector $v \in \mathbb{R}^n$, and define the directional derivative along v by

$$D_v f(x) := \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t} \quad (x \in \mathbb{R}^n)$$

whenever this limit exists.
Claim 1: The limit $D_v f(x)$ exists for $\mathcal{L}^n$-almost every x.
Because f is continuous, define the upper and lower directional derivatives by

$$\overline{D}_v f(x) := \limsup_{t \to 0} \frac{f(x + tv) - f(x)}{t} = \lim_{k \to \infty} \sup_{0 < |t| < 1/k,\, t \in \mathbb{Q}} \frac{f(x + tv) - f(x)}{t}$$

and

$$\underline{D}_v f(x) := \liminf_{t \to 0} \frac{f(x + tv) - f(x)}{t}$$

Each of these functions is Borel measurable. Thus, the set where the directional derivative fails to exist,

$$A_v := \{x \in \mathbb{R}^n : \overline{D}_v f(x) > \underline{D}_v f(x)\}$$

is Borel measurable.
Now, fix any $x, v \in \mathbb{R}^n$ and $|v| = 1$ as before, define the function $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(t) := f(x + tv), \quad t \in \mathbb{R}$$

Since f is Lipschitz, φ inherits this property and is hence absolutely continuous. Thus φ is differentiable $\mathcal{L}^1$ almost everywhere. Thus, for any line L parallel to v, we have that $\mathcal{H}^\infty(A_v \cap L) = 0$. By applying Fubini's Theorem, we conclude that $\mathcal{L}^n(A_v) = 0$. Proving Claim 1.
It follows from Claim 1 that

$$\operatorname{grad} f(x) := (f_{x_1}(x), \dots, f_{x_n}(x))$$

exists for $\mathcal{L}^n$-almost every x.
Claim 2: For $\mathcal{L}^n$-almost every x, one has

$$D_v f(x) = v \cdot \operatorname{grad} f(x)$$

We let $v = (v_1, \dots, v_n)$ and $\zeta \in C_c^\infty(\mathbb{R}^n)$. Then

$$\int_{\mathbb{R}^n} \left(\frac{f(x + tv) - f(x)}{t} \right) \zeta(x)\, dx = - \int_{\mathbb{R}^n} f(x) \left(\frac{\zeta(x) - \zeta(x - tv)}{t} \right) dx$$

We substitute $t = 1/c$ so that

$$\left|\frac{f(x+\frac{1}{c}v) - f(x)}{1/c}\right| \leq K|v| = K, \quad \text{where } K \text{ is the Lipschitz constant of } f$$

which, by the Dominated Convergence Theorem, gives

$$\int_{\mathbb{R}^n} D_v f(x)\, \zeta(x)\, dx = -\int_{\mathbb{R}^n} f(x) D_v \zeta(x)\, dx.$$

$$= -\sum_{i=1}^{n} v_i \int_{\mathbb{R}^n} f(x)\zeta_{x_i}(x)\, dx = \sum_{i=1}^{n} v_i \int_{\mathbb{R}^n} \partial f_{x_i}(x)\zeta(x)\, dx = \int_{\mathbb{R}^n} (v \cdot \operatorname{grad} f(x))\, \zeta(x)\, dx.$$

where we use Fubini's Theorem and the absolute continuity of f on lines. Since $\zeta \in C_c(\mathbb{R}^\ltimes)$ is arbitrary, it follows that $D_v f(x) = v \cdot \operatorname{grad} f(x)$ $\mathcal{L}^n$ almost everywhere. Now select a countable dense subset $\{v_k\}_{k=1}^{\infty}$ of $\partial B(1)$ where $B(1)$ is the closed ball with centre 1. For each k, define

$$A_k := \{x \in \mathbb{R}^n : D_{v_k} f(x) \text{ and } \operatorname{grad} f(x) \text{ exist and } D_{v_k} f(x) = v_k \cdot \operatorname{grad} f(x)\}$$

Let

$$A := \bigcap_{k=1}^{\infty} A_k$$

We note that $\mathcal{L}^n(\mathbb{R}^n \setminus A) = 0$.
Claim 3: For every $x \in A$, the function f is differentiable.
Fix any $x \in A$. For any $v \in \partial B(1)$, $0 \neq t \in \mathbb{R}$, define

$$Q(x, v, t) := \frac{f(x+tv) - f(x)}{t} - v \cdot \operatorname{grad} f(x)$$

Given any $v' \in \partial B(1)$, we get

$$|Q(x,v,t) - Q(x,v',t)| \leq \left|\frac{f(x+tv) - f(x+tv')}{t}\right| + |v - v'| \cdot |\operatorname{grad} f(x)|$$

$$\leq K|v - v'| + |\operatorname{grad} f(x)||v - v'| \leq (\sqrt{n}+1)K|v - v'|. \quad (*)$$

We fix $\epsilon > 0$ and choose N sufficiently large so for $v \in \partial B(1)$ then

$$|v - v_k| \leq \frac{\epsilon}{2(\sqrt{n}+1)K} \quad (**)$$

for some $k \in \{1, 2, ..., N\}$. Moreover, for each v_k, the convergence

$$\lim_{t \to 0} Q(x, v_k, t) = 0$$

allows us to choose $\delta > 0$ so that

$$|Q(x, v_k, t)| < \varepsilon/2 \tag{***}$$

for all $0 < |t| < \delta$ and $1 \leq k \leq N$. Thus, for any $v \in \partial B(1)$ and $0 < |t| < \delta$, we obtain

$$|Q(x, v, t)| \leq |Q(x, v_k, t)| + |Q(x, v, t) - Q(x, v_k, t)| < \varepsilon$$

by (*), (**), (***). Given any $y \in \mathbb{R}^n$, $y \neq x$, let $v := \frac{y-x}{|y-x|}$ such that $y = x + tv$ for $t = |x - y|$.
Then,

$$f(y) - f(x) - \operatorname{grad} f(x) \cdot (y - x) = f(x + tv) - f(x) - tv \cdot \operatorname{grad} f(x) = o(t) = o(|y - x|),$$

as $y \to x$.
This verifies that f is differentiable at x with differential $Df(x) = \operatorname{grad} f(x)$. □

2.8 Lipschitz Continuity in Banach Spaces

Definition 2.27. Let (X, d) be a complete metric space, The map $T : X \to X$ is called a *contraction mapping* if there exists $0 \leq \lambda < 1$ such that

$$d(T(x), T(y)) \leq \lambda d(x, y) \quad \text{for all } x, y \in X.$$

Remark 2.28. A contraction mapping is a special case of a Lipschitz continuous function with Lipschitz constant $\lambda < 1$. The Banach Fixed Point Theorem thus illustrates a powerful application of Lipschitz functions in guaranteeing the existence and uniqueness of fixed points.

Theorem 2.29. *Let (X, d) be a complete metric space, and let $T : X \to X$ be a Lipschitz continuous mapping with Lipschitz constant $K < 1$. Then T has a unique fixed point $x^* \in X$ such that $T(x^*) = x^*$.*

Proof. Pick any $x_0 \in X$, define $x_1 = T(x_0)$, and let $x_{n+1} = T(x_n)$. The sequence (x_n) is Cauchy.
Then, for any $k, n \geq 1$,

$$d(x_n, x_{n+k}) \leq d(x_n, x_{n+1}) + \cdots + d(x_{n+k-1}, x_{n+k}) = \sum_{i=0}^{k-1} d(x_{n+i}, x_{n+i+1}) = \sum_{i=0}^{k-1} d(f^{n+i}(x_1), f^{n+i}(x_0))$$

Using the fact that T is contractive, we have

$$\sum_{i=0}^{k-1} d(T^{n+i}(x_1), T^{n+i}(x_0)) \leq \sum_{i=0}^{k-1} K^{n+i} d(x_1, x_0) \leq K^n d(x_1, x_0) \sum_{i=0}^{\infty} K^i = \frac{K^n d(x_1, x_0)}{1 - K}$$

For all $\varepsilon > 0$, we can find $N > 0$ such that

$$\frac{K^N d(x_1, x_0)}{1-K} < \varepsilon$$

Then for any $m \geq n \geq N$

$$d(x_m, x_n) \leq \frac{K^n d(x_1, x_0)}{1-K} \leq \frac{K^N d(x_1, x_0)}{1-K} < \varepsilon$$

Therefore (x_n) is a Cauchy sequence and converges to some $x* \in X$. T is continuous, so we have

$$T(x*) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = x*$$

To prove uniqueness let x and y be fixed points, then

$$d(x, y) = d(T(x), T(y)) \leq K d(x, y)$$

since $0 \leq K < 1$ which implies $d(x, y) = 0$, so $x = y$. □

Example 2.30. Consider the function $T : [0, 1] \to [0, 1]$ defined by $T(x) = \frac{1}{2}x$. The function T is Lipschitz continuous with Lipschitz constant $K = \frac{1}{2}$. Since $K < 1$, T has a unique fixed point. Solving $T(x) = x$, we find $x^* = 0$.

Definition 2.31. A *normed vector space* is a pair $(X, \|\cdot\|)$, where X is a vector space and $\|\cdot\|$ is a map

$$\|\cdot\| : X \to \mathbb{R}, \quad x \mapsto \|x\|$$

satisfying the following axioms:

1. $\|x\| = 0 \iff x = 0$.
2. $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in K$, $x \in X$.
3. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

The map $\|\cdot\|$ is called a norm on X and $\|x\|$ is called the norm of x.

Definition 2.32. A *Banach space* is a normed vector space $(X, \|\cdot\|)$ that is complete.

Below is the revised version of the theorem and its proof using the notation K consistently:

—

Theorem 2.33. *Let E be a Banach space, $\Omega \subseteq E$ an open set, and let $I : \Omega \to E$ denote the identity map. Let $g : \Omega \to E$ be a Lipschitz continuous function with Lipschitz constant $K < 1$. Define the mapping $f = I + g$ and consider the set $\Omega' = f(\Omega)$. We aim to prove:*

1. *Ω' is open, and for any $x \in \Omega$ with $\overline{B(x,r)} \subseteq \Omega$, we have $B(f(x), r(1-K)) \subseteq \Omega'$.*

2. *f is a bi-Lipschitz homeomorphism from Ω onto Ω'.*

3. *The inverse f^{-1} is of the form $I + h$, where $h : \Omega' \to E$ is Lipschitz with constant $\frac{K}{1-K}$.*

Proof. **(1)** Let $x \in \Omega$ and suppose that $\overline{B(x,r)} \subseteq \Omega$ for some $r > 0$. We will show that

$$B(f(x), r(1-K)) \subseteq \Omega'$$

Let $y \in B(f(x), r(1-K))$. Then

$$\|y - f(x)\| < r(1-K)$$

Define the mapping $T : \overline{B(x,r)} \to E$ by

$$T(z) = y - g(z)$$

We first show that T maps $\overline{B(x,r)}$ into itself. For any $z \in \overline{B(x,r)}$,

$$\begin{aligned}\|T(z) - x\| &= \|y - g(z) - x\| \\ &= \|[y - f(x)] + [f(x) - x - g(z)]\|\end{aligned}$$

Since $f(x) = x + g(x)$, it follows that $f(x) - x = g(x)$. Hence,

$$\|T(z) - x\| = \|y - f(x) + g(x) - g(z)\| \leq \|y - f(x)\| + \|g(x) - g(z)\|$$

Using the Lipschitz property of g,

$$\|T(z) - x\| \leq r(1-K) + K\|x - z\| \leq r(1-K) + Kr = r$$

Thus, $T(z) \in \overline{B(x,r)}$.
Next, for any $z_1, z_2 \in \overline{B(x,r)}$,

$$\|T(z_1) - T(z_2)\| = \|g(z_2) - g(z_1)\| \leq K\|z_2 - z_1\|$$

Since $K < 1$, T is a contraction on the complete metric space $\overline{B(x,r)}$. By the Banach Fixed Point Theorem, there exists a unique $z \in \overline{B(x,r)}$ such that

$$z = T(z) = y - g(z)$$

It follows that

$$y = z + g(z) = f(z)$$

so $y \in \Omega'$. As y was an arbitrary point in $B(f(x), r(1-K))$, we conclude that

$$B(f(x), r(1-K)) \subseteq \Omega'$$

and hence Ω' is open.

(2) To show that f is a bi-Lipschitz homeomorphism, we first prove that f is injective. Suppose that $f(x_1) = f(x_2)$ for some $x_1, x_2 \in \Omega$. Then,

$$x_1 + g(x_1) = x_2 + g(x_2)$$

so that

$$x_1 - x_2 = g(x_2) - g(x_1)$$

Taking norms yields

$$\|x_1 - x_2\| = \|g(x_2) - g(x_1)\| \leq K\|x_1 - x_2\|$$

Since $K < 1$, it must be that $\|x_1 - x_2\| = 0$, hence $x_1 = x_2$.

Next, for any $x_1, x_2 \in \Omega$,

$$\begin{aligned}
\|f(x_1) - f(x_2)\| &= \|(x_1 + g(x_1)) - (x_2 + g(x_2))\| \\
&= \|x_1 - x_2 + g(x_1) - g(x_2)\| \\
&\leq \|x_1 - x_2\| + \|g(x_1) - g(x_2)\| \\
&\leq \|x_1 - x_2\| + K\|x_1 - x_2\| \\
&\leq (1 + K)\|x_1 - x_2\|
\end{aligned}$$

Thus, f is Lipschitz with constant $1 + K$.

Furthermore, using the reverse triangle inequality, we have:

$$\begin{aligned}
\|f(x_1) - f(x_2)\| &= \|x_1 - x_2 + g(x_1) - g(x_2)\| \\
&\geq \|x_1 - x_2\| - \|g(x_1) - g(x_2)\| \\
&\geq \|x_1 - x_2\| - K\|x_1 - x_2\| \\
&= (1 - K)\|x_1 - x_2\|
\end{aligned}$$

Thus

$$\|x_1 - x_2\| \leq \frac{1}{1 - K}\|f(x_1) - f(x_2)\|$$

This shows that the inverse mapping f^{-1} is Lipschitz with constant $\frac{1}{1-K}$.

Since f is continuous as the sum of continuous functions, injective, and both f and f^{-1} are Lipschitz, f is a bi-Lipschitz homeomorphism from Ω onto Ω'.

(3) Finally, we express the inverse mapping in the form $I + h$. For $y \in \Omega'$, let $z = f^{-1}(y)$, so that

$$y = f(z) = z + g(z)$$

Then,

$$z = y - g(z)$$

or equivalently,

$$f^{-1}(y) = y - g(f^{-1}(y))$$

where

$$h(y) = -g(f^{-1}(y))$$

Thus, we have

$$f^{-1}(y) = y + h(y)$$

For any $y_1, y_2 \in \Omega'$ and note that

$$\begin{aligned}\|h(y_1) - h(y_2)\| &= \|g(f^{-1}(y_2)) - g(f^{-1}(y_1))\| \\ &\leq K\|f^{-1}(y_2) - f^{-1}(y_1)\| \\ &\leq \frac{K}{1-K}\|y_1 - y_2\|\end{aligned}$$

This proves that h is Lipschitz with constant $\frac{K}{1-K}$.

□

3 Knots

3.1 Introduction

Knots are fundamental objects in topology and geometry, representing embeddings of circles into three-dimensional spaces. In real algebraic geometry, knots can be studied as real algebraic varieties, offering an algebraic perspective on their topological properties. Lipschitz methods also offer a toolset for studying knots. They allow for the quantification of the geometry of knots, particularly with the use of bi-Lipschitz equivalence.

3.2 Basic Concepts in Knot Theory

Definition 3.1. A *knot* is a smooth embedding of the circle S^1 into $\mathbb{R}^3$, denoted as $P : S^1 \hookrightarrow S^3$.

Definition 3.2. A *link* L, is a finite union of pairwise nonintersecting knots. The number of knots in the union comprising L is called the number of components of the link. A *knot* is a link with one component.

Definition 3.3. Two mathematical knots are *equivalent* if one can be transformed into the other. These transformations correspond to manipulations of a knotted string that do not involve cutting it or passing it through itself.

Definition 3.4. Two links L_1, L_2 in S^3 are equivalent if there exists an orientation-preserving homeomorphism $f : S^3 \to S^3$ such that $f(L_1) = L_2$.

This idea can be formalised through the notion of an isotopy.

Definition 3.5. A continuous family of homeomorphisms $\{H_t : \mathbb{R}^3 \to \mathbb{R}^3\}_{t\in[0,1]}$ is called an ambient isotopy if

1. H_0 is the identity map on $\mathbb{R}^3$
2. For each $t \in [0, 1]$, H_t is a homeomorphism
3. The map $(x, t) \mapsto H_t(x)$ is continuous
4. $H_1 \circ k_0 = k_1$

Definition 3.6. Two links L_1, L_2 in S^3 are equivalent if there exists an ambient isotopy between L_1 and L_2.

3.3 Diagrams and Projections

Working directly with knots in $\mathbb{R}^3$ can be challenging to visualise. To simplify matters, we use *knot diagrams*, which are planar representations of knots obtained via projection.

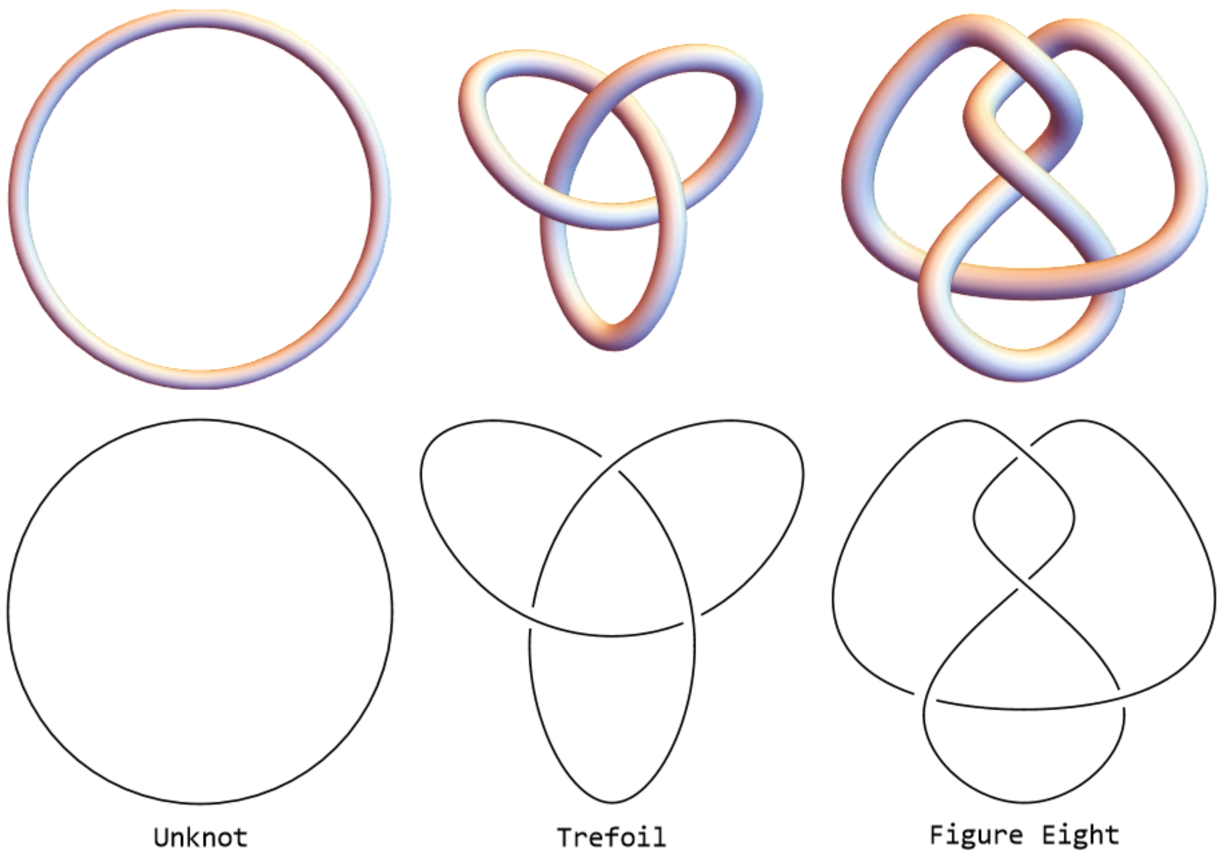
Definition 3.7. A projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ is *regular* for a knot k if the following conditions are satisfied:

1. $\pi \circ k$ is an immersion except at a finite number of points called *crossings*.
2. The preimages of the crossing point at each crossing are transverse intersections.
3. There are no triple or higher-order crossings.

A *crossing* in a knot diagram is a place where the knot curve crosses, going over or under – itself.

3.4 Simplest Knots

Below are images of the three simplest knots, which are those made with the least number of crossings, along with their respective knot diagram.



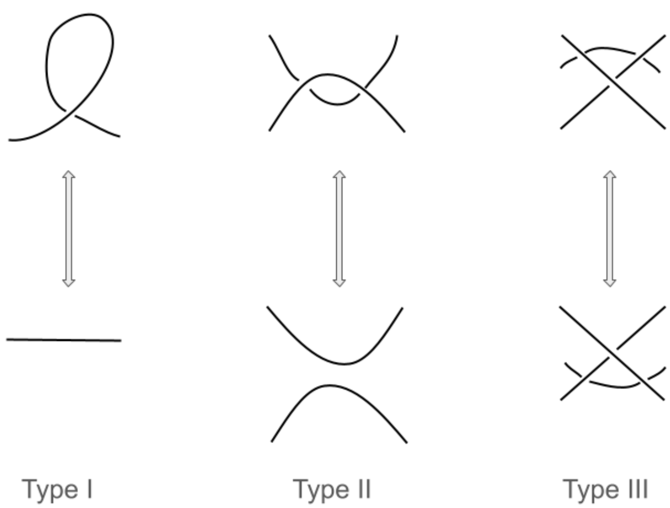
3.5 Reidemeister Moves

The Reidemeister moves are local transformations on knot diagrams that correspond to ambient isotopies of knots in $\mathbb{R}^3$. They are fundamental in knot theory because they provide a way to manipulate knot diagrams while preserving the knot type.

Type I Move (Twist/Untwist): Adds or removes a twist in a single strand.

Type II Move (Poke): Moves one strand completely over another, introducing or removing two crossings.

Type III Move (Slide): Slides a strand over or under a crossing between two other strands, changing the ordering at the crossing without introducing or removing crossings.

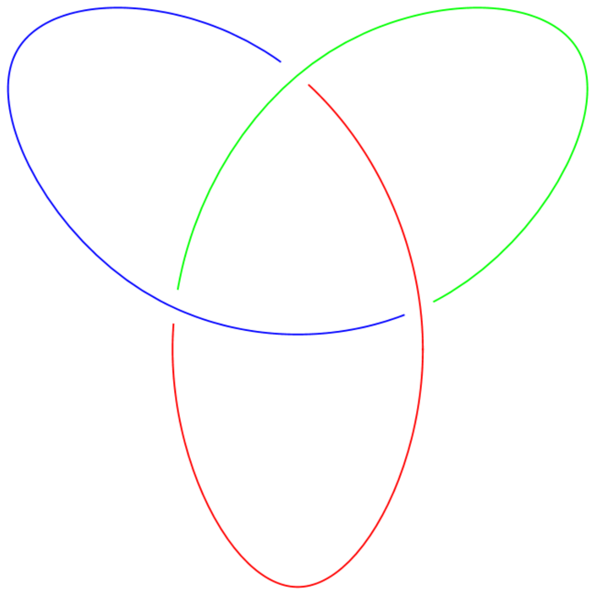


Theorem 3.8. *Two knot diagrams represent ambient isotopic knots if and only if they are related by a finite sequence of Reidemeister moves of types I, II, and III.*

Definition 3.9. A projection of a knot or link is *tricolourable* if every arc of the knot can be coloured with one of three different colours such that:

1. At each crossing, either arcs of all three colours meet, or only arcs of a single colour meet.
2. At least two colours are used.

Remark 3.10. Tricolourability is invariant under Reidemeister moves. In other words, if a given diagram of a knot is tricolourable, then every diagram of the same knot is tricolourable



The above diagram shows the tricolourability of the trefoil knot.

4 Semialgebraic Sets

4.1 Semialgebraic maps

Definition 4.1. A set $X \subset \mathbb{R}^n$ is called an *algebraic set* if there exists polynomial functions $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, k$, such that

$$X = \{x \in \mathbb{R}^n \mid f_1(x) = 0, \ldots, f_k(x) = 0\}$$

Definition 4.2. A subset $X \subset \mathbb{R}^n$ is called a *semialgebraic set* if there exists polynomial functions $f_{i,j}, g_{i,j} : \mathbb{R}^n \to \mathbb{R}$ for $1 \leq i \leq p$ and $1 \leq j \leq q$, such that

$$X = \bigcup_{i=1}^{p} \bigcap_{j=1}^{q} \{x \in \mathbb{R}^n \mid f_{i,j}(x) = 0,\ g_{i,j}(x) > 0\}$$

Definition 4.3. Let $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ be two semi-algebraic sets. A mapping $f : A \to B$ is *semi-algebraic* if its graph

$$\mathrm{Graph}(f) = \{(x, f(x)) \mid x \in A\} \subset \mathbb{R}^{m+n}$$

is a semi-algebraic set in $\mathbb{R}^{m+n}$.

Example 4.4. The absolute value is semialgebraic.

Remark 4.5. The following are properties of semialgebraic sets.

1. An algebraic set is semialgebraic.
2. Semialgebraic sets are finite unions of intervals and points in $\mathbb{R}$.
3. If $X, Y \subset \mathbb{R}^n$ are semialgebraic sets, then $X \cup Y$, $X - Y$, and $X \cap Y$ are semialgebraic.
4. If $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ are semialgebraic sets, then $Z \subset \mathbb{R}^n \times \mathbb{R}^m$ is a semialgebraic set.

Theorem 4.6. *Let x and y be two points of $\mathbb{R}^{>}$, U an open semi-algebraic set containing the segment $[x, y]$, and let $f : U \to \mathbb{R}^l$ be a semi-algebraic function whose first order derivative exist. Then*

$$\|f(x) - f(y)\| \leq K\|x - y\|,$$

where $K = \sup\{\|df_z\| \mid z \in [x, y]\}$

Proof. Let

$$g(t) = f((1-t)x + ty) \quad \text{for } t \in [0, 1]$$

Then

$$\|g'(t)\| \leq K\|x - y\| \quad \text{for } t \in [0, 1]$$

Let $c > 0$ and define

$$A_c = \{t \in [0,1] \mid \|g(t) - g(0)\| \leq K\|x - y\|t + ct\}$$

This is a closed semi-algebraic subset of $[0,1]$ containing 0. It contains a largest element t_0. Suppose $t_0 \neq 1$. Then

$$\|g(t_0) - g(0)\| \leq K\|x - y\|t_0 + ct_0$$

Since $\|g'(t_0)\| \leq K\|x - y\|$, we can find $\mu > 0$ in $\mathbb{R}$ such that, if $t_0 < t < t_0 + \mu$, then

$$\|g(t) - g(t_0)\| \leq K\|x - y\|(t - t_0) + c(t - t_0)$$

Thus, for $t_0 < t < t_0 + \mu$, we have

$$\|g(t) - g(0)\| \leq K\|x - y\|t + ct$$

which contradicts the maximality of t_0. Thus, $1 \in A_c$ for every c, which implies the conclusion of the theorem. □

Definition 4.7. An *arc* in $\mathbb{R}^n$ is a germ at the origin of a semialgebraic mapping $\gamma : [0, \varepsilon) \to \mathbb{R}^n$ such that $\|\gamma(t)\| = t$.

4.2 Lipschitz Normal Embeddings

Definition 4.8. Let X be a subset of $\mathbb{R}^n$. We define the following metrics on X:

1. The *outer metric*:
$$d_{\text{out}}(x, y) = \|x - y\|$$
which is the Euclidean metric induced on X.

2. The *inner metric*:
$$d_{in}(x, y) = \inf_{\gamma \in \Gamma(x,y)} l(\gamma)$$
where $\Gamma(x, y)$ is the set of rectifiable arcs $\gamma : [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$, and $l(\gamma)$ is the length of γ. This is the length of the shortest path in X connecting x and y

We have that $d_{\text{out}}(x, y) \leq d_{in}(x, y)$.

Definition 4.9. A semialgebraic set X is called *Lipschitz Normally Embedded (LNE)* if the inner and outer metrics on X are equivalent. That is, there exists a constant $K > 0$ such that:

$$d_{\text{out}}(x, y) \leq d_{in}(x, y) \leq K\, d_{\text{out}}(x, y)$$

for all $x, y \in X$.

Definition 4.10. A space X is *locally Lipschitz normally embedded* at $x \in X$ if there exists an open neighborhood U of x such that U is Lipschitz normally embedded. We say that X is *locally Lipschitz normally embedded* if this condition holds for all $x \in X$.

Example 4.11. An example of a set that is not Lipschitz normally embedded is the real curve defined by $x^3 - y^2 = 0$. In this case,

$$d_{\text{out}}((t^2, t^3), (t^2, -t^3)) = 2|t|^3$$

but

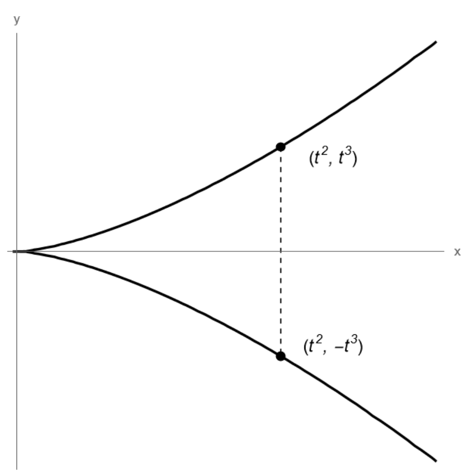
$$d_{in}((t^2, t^3), (t^2, -t^3)) = 2|t|^2 + o(t^2)$$

Then,

$$\lim_{t \to 0} \frac{d_{in}((t^2, t^3), (t^2, -t^3))}{d_{\text{out}}((t^2, t^3), (t^2, -t^3))} = +\infty$$

and hence there cannot exist a constant $K > 0$ satisfying

$$d_{in}(x, y) \leq K\, d_{\text{out}}(x, y)$$



$x^3 - y^2 = 0$

Theorem 4.12. *Let X be a connected, compact locally Lipschitz normally embedded space. Then X is Lipschitz normally embedded.*

Proof. For every $x \in X$, we let U_x be a Lipschitz normally embedded neighbourhood of x. Also let K_x be a bi-Lipschitz constant. Thus if $y \in X$ is very close to x, then we have

$$d_{in}(x, y) \leq K_x \, d_{\text{out}}(x, y)$$

Now consider the map

$$f(x, y) := \frac{d_{in}(x, y)}{d_{\text{out}}(x, y)} : M \times M \to \mathbb{R}$$

Let $U \subset M \times M$ be a small open tubular neighbourhood of the diagonal. This expression implies that f is continuous on the compact set $(M \times M) \setminus U$ and is locally bounded at every point. Therefore, f is globally bounded on $(M \times M) \setminus U$ and on U as well. □

Theorem 4.13. *Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$, and let $Z = X \times Y \subset \mathbb{R}^{n+m}$. Z is Lipschitz normally embedded if and only if X and Y are Lipschitz normally embedded.*

Proof. ($\Rightarrow$): Assume X and Y are LNE with constants K_X and K_Y, respectively. We show Z is LNE. For points $(x_1, y_1), (x_2, y_2) \in Z$, the intrinsic distance in Z satisfies:

$$d_{in}^Z\left((x_1, y_1), (x_2, y_2)\right) \leq d_{in}^Z\left((x_1, y_1), (x_1, y_2)\right) + d_{in}^Z\left((x_1, y_2), (x_2, y_2)\right)$$

Restricting paths to slices $\{x_1\} \times Y$ and $X \times \{y_2\}$, we bound each term

$$d_{in}^Z\left((x_1, y_1), (x_1, y_2)\right) \leq d_{in}^Y(y_1, y_2) \leq K_Y \|y_1 - y_2\|$$

$$d_{in}^Z\left((x_1, y_2), (x_2, y_2)\right) \leq d_{in}^X(x_1, x_2) \leq K_X \|x_1 - x_2\|$$

Thus

$$d_{in}^Z((x_1, y_1), (x_2, y_2)) \leq K_Y d_{\text{out}}^Y(y_1, y_2) + K_X d_{\text{out}}^X(x_1, x_2)$$

By definition

$$d_{\text{out}}^Z((x_1, y_1), (x_2, y_2))^2 = d_{\text{out}}^Y(y_1, y_2)^2 + d_{\text{out}}^X(x_1, x_2)^2$$

Therefore,

$$d_{\text{out}}^Z((x_1, y_1), (x_1, y_2)) \leq d_{\text{out}}^Z((x_1, y_1), (x_2, y_2))$$

and

$$d_{\text{out}}^Z((x_1, y_2), (x_2, y_2)) \leq d_{\text{out}}^Z((x_1, y_1), (x_2, y_2))$$

Combining these we get

$$d_{in}^Z((x_1, y_1), (x_2, y_2)) \leq (K_Y + K_X) d_{\text{out}}^Z((x_1, y_1), (x_2, y_2))$$

as required.

($\Leftarrow$): For the other direction, we let $p, q \in X$, and consider any path $\gamma : [0,1] \to Z$ such that $\gamma(0) = (p, 0)$ and $\gamma(1) = (q, 0)$. Here $\gamma(t) = (\gamma_X(t), \gamma_Y(t))$, where $\gamma_X : [0,1] \to X$ and $\gamma_Y : [0,1] \to Y$ are paths with $\gamma_X(0) = p$ and $\gamma_X(1) = q$. Since $l(\gamma) \geq l(\gamma_X)$, we have

$$d_{in}^X(p,q) \leq d_{in}^Z((p,0),(q,0))$$

Z is Lipschitz normally embedded so there exists a constant $K > 1$ such that

$$d_{in}^Z(z_1,z_2) \leq K\, d_{\text{out}}^Z(z_1,z_2)$$

for all $z_1, z_2 \in Z$. Also $d_{\text{out}}^Z((p,0),(q,0)) = d_{\text{out}}^X(p,q)$ as X is embedded in Z as $X \times \{0\}$, so

$$d_{in}^X(p,q) \leq K\, d_{\text{out}}^X(p,q)$$

The same argument applies to Y. $\square$

5 β-Horns

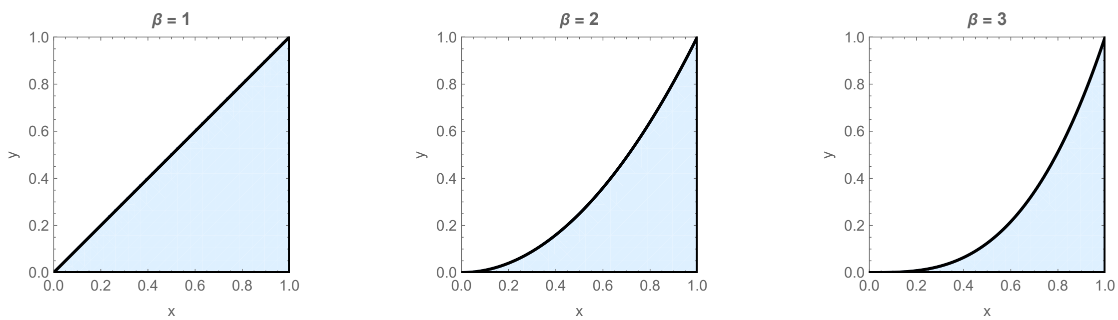
In the study of real algebraic geometry, β-Hölder triangles and β-Horns play a crucial role in understanding the local structure of semialgebraic sets with controlled Hölder regularity. A standard β-Horn is constructed as the union of semialgebraic β-Hölder triangles sharing the same principal vertex. Formally, we write

$$H_\beta = \bigcup_i T_i^\beta$$

where each triangle T_i^β is bounded by two boundary curves.

Definition 5.1. A standard β-Hölder triangle ($\beta \geq 1$) is defined as

$$T_\beta = \{(x, y) \in \mathbb{R}^2 \mid y \leq x^\beta,\ y \geq 0,\ 0 \leq x \leq 1\}$$



β-Hölder triangles with varying values for β.

Theorem 5.2. *A standard β-Hölder triangle ($\beta \geq 1$) is normally embedded.*

Proof. Let $p = (x_p, y_p)$ and $q = (x_q, y_q)$ be two arbitrary points in T_β. By definition of T_β, we have

$$0 \leq x_p \leq 1, \quad 0 \leq x_q \leq 1, \quad y_p \leq x_p^\beta, \quad y_q \leq x_q^\beta.$$

Without loss of generality, assume $x_p \leq x_q$. To connect p and q, construct a simple three segment path γ that lies entirely in T_β: 1. Move from (x_p, y_p) to $(x_p, 0)$, 2. Move from $(x_p, 0)$ to $(x_q, 0)$, 3. Move from $(x_q, 0)$ to (x_q, y_q).
The total length of this path is:

$$\ell(\gamma) = |y_p| + |x_q - x_p| + |y_q| = y_p + (x_q - x_p) + y_q$$

Since T_β includes all (x, y) such that $0 \leq y \leq x^\beta$, this path is contained entirely within T_β. Using the fact that $y_p \leq x_p^\beta$ and $y_q \leq x_q^\beta$, we bound $\ell(\gamma)$ as:

$$\ell(\gamma) \leq x_p^\beta + (x_q - x_p) + x_q^\beta$$

Since $x_p, x_q \in [0, 1]$ and $\beta \geq 1$, we know that

$$x_p^\beta \leq 1 \quad \text{and} \quad x_q^\beta \leq 1$$

Thus $\ell(\gamma) \leq 2 + (x_q - x_p)$ We have that $d_{\text{out}}(p,q) = \sqrt{(x_q - x_p)^2 + (y_q - y_p)^2}$ Therefore, clearly $x_q - x_p \leq d_{\text{out}}(p,q)$ Using this, we have that

$$\ell(\gamma) \leq 2 + d_{\text{out}}(p,q)$$

We require

$$\ell(\gamma) \leq K \cdot d_{\text{out}}(p,q)$$

which gives

$$2 + d_{\text{out}}(p,q) \leq K \cdot d_{\text{out}}(p,q)$$

for K large enough. In other words we choose K such that

$$K \geq \frac{2}{d_{\text{out}}(p,q)} + 1$$

and thus the inequality is satisfied. Since the path γ lies entirely within T_β, we conclude that

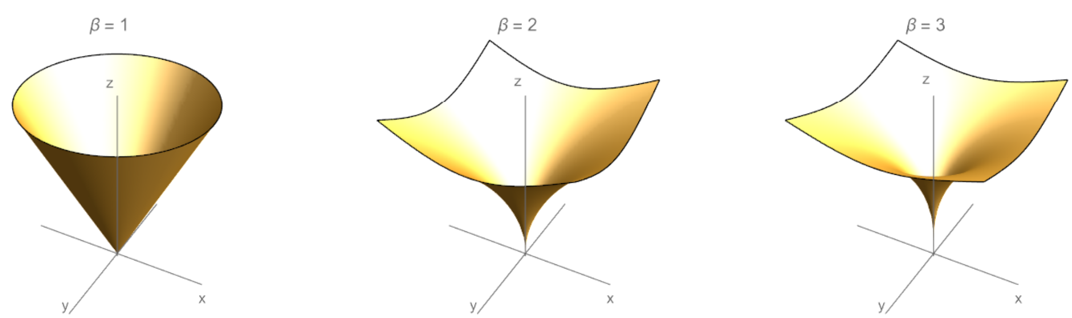
$$d_{\text{in}}(p,q) \leq \ell(\gamma) \leq K \cdot d_{\text{out}}(p,q)$$

Therefore T_β is Lipschitz normally embedded. □

Definition 5.3. The semialgebraic surface in $\mathbb{R}^3$ defined by

$$X = \{(x,y,z) \mid (x^2 + y^2)^q = z^{2p},\ z \geq 0\},$$

with $\beta = \frac{p}{q} \geq 1$, is known as a β-Horn.



β-Horns with varying values for β.

5.1 E_8 singularity

Remark 5.4. The E_8 singularity is an isolated singularity in real algebraic geometry, given by the equation

$$x^3 + y^5 + z^2 = 0$$

It cannot be decomposed into simpler singularities and plays a key role in understanding the local structure of real algebraic surfaces.

Example 5.5. Let $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x, y, z) = x^2 - y^3 + z^5$, and X be the kernel of f i.e.

$$X = \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = 0\}.$$

Let $\gamma^{\pm}(t) = (\pm t^{3/2}, t, 0)$, $t \in \mathbb{R}$, be the two branches of the curve $X \cap \{z = 0\}$, and let β be the curve defined by $X \cap \{x = 0\}$. At $t = 0$ the common unit tangent vector to γ^+ and γ^- is $(0, 1, 0)$. The tangent to the curve β at the origin is $(0, 0, 1)$.
For any fixed t, and any arc C in X from $\gamma^-(t)$ to $\gamma^+(t)$, then C always intersects β. Let this intersection point be $Q = (0, \bar{y}, \bar{z})$ and we can assume Q is unique. We have

$$d_{in}(\gamma^-(t), \gamma^+(t)) = \inf_C l(C),$$

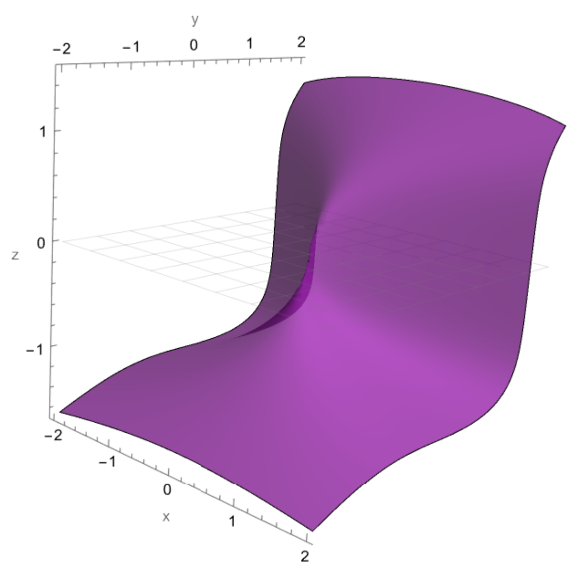
where $l(C) = l(C^-) + l(C^+)$, and C^- and C^+ are the arcs from $\gamma^-(t)$ and $\gamma^+(t)$ to Q, respectively. Then we can see that

$$l(C) \geq \|\gamma^-(t) - Q\| + \|\gamma^+(t) - Q\| = 2\sqrt{t^3 + (t - \bar{y})^2 + \bar{z}^5} \approx t,$$

for $\bar{y}, \bar{z}$ sufficiently small. We can then use the same argument that we used to show that $x^2 - y^3 = 0$ is not locally normally embedded at the origin. This is

$$\frac{d_{in}(\gamma^-(t), \gamma^+(t))}{d_{\text{out}}(\gamma^-(t), \gamma^+(t))} \to \infty$$

as $t \to 0$. The result then follows.



Plot of $x^2 - y^3 + z^5 = 0$

5.2 Distortion

Let γ be an embedded rectifiable closed curve in $\mathbb{R}^n$ of finite length $L(\gamma)$. We can then assume that $\gamma : \mathbb{R}/L\mathbb{Z} \to \mathbb{R}^n$ is a Lipschitz parameterisation by arclength. We also let it be that $L = 2\pi$.
For $x, y \in \gamma$ as before $d_{out}(x, y)$ is the straight-line distance between them in $\mathbb{R}^n$, and $d_{in}(x, y)$ the arclength distance between these two points along γ. This arclength distance is always the shorter way around γ, so we have that $d_{in}(x, y) \ngtr L/2$. Given any point p on γ, there exists an opposite point p^*such that $d_{in}(p, p^*) = L/2$.
Two points x, y along a knot K separate K into two complementary arcs, γ_{xy} (from x to y) and γ_{yx}. We let ℓ_{xy} denote the length of γ_{xy}. Distortion contrasts the shorter arclength distance

$$d_{in}(x, y) := \min(\ell_{xy}, \ell_{yx}) \leq \frac{\ell(K)}{2}$$

with the straight-line distance $|x - y|$ in $\mathbb{R}^3$.

Definition 5.6. The *distortion* between distinct points x and y on a curve γ is

$$\delta(x, y) := \frac{d_{in}(x, y)}{d_{out}(x, y)} \geq 1$$

This quantity is a Lipschitz constant for the inverse of the map γ.
The *distortion* of γ is the supremum

$$\delta(\gamma) := \sup \delta(x, y)$$

taken over all pairs of distinct points.

Theorem 5.7. *Given any closed rectifiable curve $\gamma \subset \mathbb{R}^n$, the distortion satisfies*

$$\delta(\gamma) \geq \frac{\pi}{2}$$

Proof. Rescale to let $L(\gamma) = 2\pi$, and parameterise the curve by arclength $s \in \mathbb{R}/2\pi\mathbb{Z}$. For the opposite points p and p^* we have arclength $d(p, p^*) = \pi$. We wish to prove that for some p, $|p - p^*| \leq 2$. Because then $\delta(p, p^*) \geq \pi/2$ and then we are done.
Consider the new curve in $\mathbb{R}^n$ defined by

$$f(s) := p - p^* = \gamma(s) - \gamma(s + \pi)$$

Using the triangle inequality $|f'(s)| \leq |\gamma'(s)| + |\gamma'(s + \pi)| = 2$ almost everywhere, so f is Lipschitz with Lipschitz constant at most two. Also $f(s + \pi) = -f(s)$ for all s.
We need to show that $|f| \leq 2$ somewhere. For sake of contradiction suppose that this is not the case. Then f lies outside the closed ball of radius two in $\mathbb{R}^n$. Then any arc of f from s to $s + \pi$ is an arc between opposite points in $\mathbb{R}^n$ that avoids this closed ball. Here, the length exceeds the distance between opposite points on a sphere of radius two, which is 2π.
But since the parameterisation of f has a constant of at most two, the length of this arc is at most 2π, a contradiction. $\square$

Example 5.8. The trefoil knot is Lipschitz normal embedded.

Proof. The trefoil knot K_T is given by a smooth embedding

$$\gamma : [0, 2\pi] \to \mathbb{R}^3$$

with parameterisation

$$\gamma(t) = (\sin(t) + 2\sin(2t), \cos(t) - 2\cos(2t), -\sin(3t)), \quad t \in [0, 2\pi].$$

The functions $\sin(nt)$ and $\cos(nt)$ are smooth, ensuring that $\gamma(t)$ is a C^∞-embedding.
For points $x = \gamma(t_1)$ and $y = \gamma(t_2)$ on the knot, the intrinsic metric $d_{in}(x, y)$ is the arc length of the curve γ between $\gamma(t_1)$ and $\gamma(t_2)$

$$d_{in}(x, y) = \int_{t_1}^{t_2} \|\gamma'(t)\| \, dt$$

Let

$$\|\gamma'(t)\|_{\min} \leq \|\gamma'(t)\| \leq \|\gamma'(t)\|_{\max}$$

Since $\|\gamma'(t)\|$ is continuous on the compact interval $[0, 2\pi]$, using Extreme Value Thereom both $\|\gamma'(t)\|_{\min} > 0$ and $\|\gamma'(t)\|_{\max} < \infty$ exist.
Consider the Euclidean distance $\|x - y\| = \|\gamma(t_1) - \gamma(t_2)\|$.
Use the Mean Value Theorem for arc length,

$$\int_{t_1}^{t_2} \|\gamma'(t)\| \, dt \geq \int_{t_1}^{t_2} \|\gamma'(t)\|_{\min} = \|\gamma'(t)\|_{\min} \cdot \int_{t_1}^{t_2} 1 \, dt$$

Thus

$$d_{in}(x, y) = \int_{t_1}^{t_2} \|\gamma'(t)\| \, dt \geq \|\gamma'(t)\|_{\min} \cdot |t_2 - t_1|$$

where $|t_2 - t_1|$ is the parameter difference.
The Mean Value Theorem for $\gamma(t)$ gives

$$\gamma(t_2) - \gamma(t_1) = \gamma'(t^) \cdot (t_2 - t_1)$$

for some $t^{\in}[t_1, t_2]$. Taking the norm on both sides yields

$$\|\gamma(t_2) - \gamma(t_1)\| = \|\gamma'(t^)\| \cdot |t_2 - t_1|$$

Since $\|\gamma'(t)\|$ is continuous, it is bounded on the interval $[t_1, t_2]$, so

$$\|\gamma'(t^)\| \leq \|\gamma'(t)\|_{\max}$$

Substituting this bound,

$$\|\gamma(t_2) - \gamma(t_1)\| \leq \|\gamma'(t)\|_{\max} \cdot |t_2 - t_1|$$

Similarly,

$$\|\gamma(t_2) - \gamma(t_1)\| \geq \|\gamma'(t)\|_{\min} \cdot |t_2 - t_1| \tag{*}$$

Thus, we establish that

$$d_{\text{out}}(x, y) = \|\gamma(t_2) - \gamma(t_1)\| \leq \|\gamma'(t)\|_{\max} \cdot |t_2 - t_1|$$

Thus

$$\frac{d_{\text{out}}(x, y)}{\|\gamma'(t)\|_{\max}} \leq |t_2 - t_1| \leq \frac{d_{in}(x, y)}{\|\gamma'(t)\|_{\min}}$$

Again, use the definition of $d_{in}(x, y)$

$$\int_{t_1}^{t_2} \|\gamma'(t)\| \, dt \leq \int_{t_1}^{t_2} \|\gamma'(t)\|_{\max} = \|\gamma'(t)\|_{\max} \cdot \int_{t_1}^{t_2} 1 \, dt$$

Thus,

$$d_{in}(x, y) = \int_{t_1}^{t_2} \|\gamma'(t)\| \, dt \leq \|\gamma'(t)\|_{\max} \cdot |t_2 - t_1|$$

Thus,

$$\frac{d_{in}(x, y)}{\|\gamma'(t)\|_{\max}}) \leq |t_2 - t_1| \tag{**}$$

We combine $(*)$ and $(**)$ to get

$$d_{in}(x, y) \leq \|\gamma'(t)\|_{\max} \cdot |t_2 - t_1| \leq \|\gamma'(t)\|_{\max} \cdot \frac{d_{\text{out}}(x, y)}{\|\gamma'(t)\|_{\min}}$$

Thus,

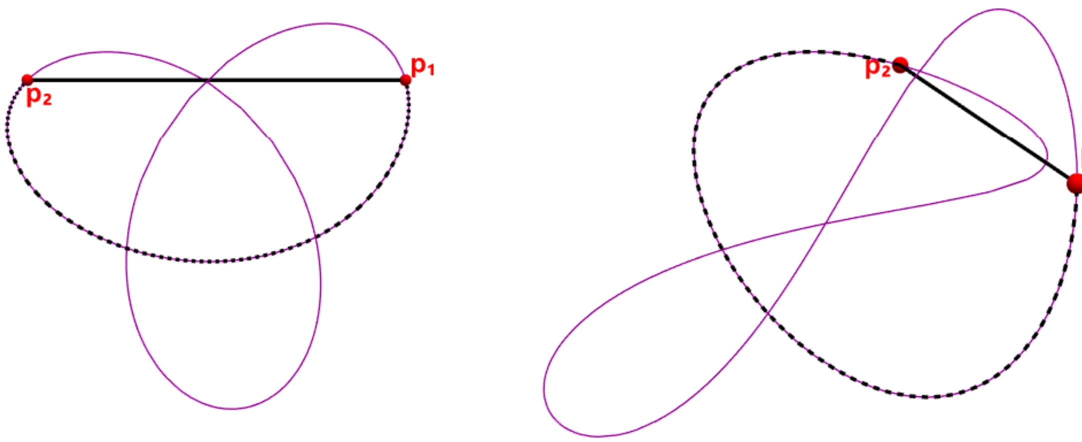
$$d_{in}(x, y) \leq \frac{\|\gamma'(t)\|_{\max}}{\|\gamma'(t)\|_{\min}} d_{\text{out}}(x, y)$$

The arc length $d_{in}(x, y)$ is bounded above by the straight-line distance multiplied by the maximum Lipschitz constant of γ.
Thus,

$$d_{in}(x, y) \leq K d_{out}(x, y)$$

where $K = \frac{\|\gamma'(t)\|_{\max}}{\|\gamma'(t)\|_{\min}}$ is the Lipschitz constant for K_T.

□



The outer (solid line) and inner (dashed line) metric between points p_1 and p_2 on the Trefoil

6 Cones

In Lipschitz geometry, cones provide a simplified model of a space near singularities, revealing key metric and topological properties. Their analysis helps characterise how spaces behave under Lipschitz transformations, especially offering insights into singularities. We now state and prove the two main results from [12]

6.1 Tangent Cones

Let $X \subseteq \mathbb{R}^m$ be a set and $p \in X$.

Definition 6.1. Consider a sequence $S = \{t_j\}_{j\in\mathbb{N}}$ of positive real numbers satisfying $\lim_{j\to\infty} t_j = 0$. A vector $v \in \mathbb{R}^m$ is called a *direction* of X at p relative to S if there exists a sequence $\{x_j\} \subseteq X \setminus \{p\}$ and $j_0 \in \mathbb{N}$ such that $\|x_j - p\| = t_j$ for all $j \geq j_0$, and

$$\lim_{j\to\infty} \frac{x_j - p}{t_j} = v$$

The collection of all such directions at p relative to S is denoted $\mathcal{D}_p^S(X)$.

Definition 6.2. We denote $Z \subset \mathbb{R}^m$ as the tangent cone of X at p if there exists a sequence $S = \{t_j\}_{j\in\mathbb{N}}$ with $t_j \to 0$ such that

$$Z = \{tv \mid v \in D_p^S(X) \text{ and } t \geq 0\}$$

When this cone is uniquely determined at p, it is called the *tangent cone* of X at p and written as T_pX.

Theorem 6.3. *Let $X \subseteq \mathbb{R}^m$, and let $x_0 \in X$. If X is Lipschitz normally embedded and has a unique tangent cone $T_{x_0}X$, then $T_{x_0}X$ is also Lipschitz normally embedded.*

Proof. Assume x_0 coincides with the origin. Let $K > 0$ be the Lipschitz constant such that

$$d_X(x, y) \leq K\|x - y\| \quad \forall x, y \in X$$

Fix $\delta > 0$ and choose $\lambda > K + 1 + \delta$. For $x, y \in X$ with $\|x\| = \|y\| = t$, any path α connecting x to y such that

$$\text{length}(\alpha) \leq d_X(x, y) + \delta\|x - y\|$$

must lie within the compact ball $B[0, \lambda t]$. This follows because paths exiting this ball have length at least $2(\lambda - 1)t$ and then

$$\text{length}(\alpha) > (K + \delta)\|x - y\|$$

Let $v, w \in T_0X$ with $\|v\| = \|w\| = 1$. We prove that

$$d_{T_0X}(v, w) \leq K\|v - w\|$$

Select sequences $\{x_n\}, \{y_n\} \subset X$ with $\|x_n\| = \|y_n\| = t_n \to 0$, where $x_n/t_n \to v$ and $y_n/t_n \to w$. For every n, we let $\gamma_n : [0,1] \to X$ be a path in X from x_n to y_n such that

$$\text{length}(\gamma_n) \leq d_X(x_n, y_n) + \delta\|x_n - y_n\|$$

Define the scaled paths $\alpha_n = \gamma_n/t_n$. Each α_n is confined to $B[0, \lambda t_n]$, and their lengths satisfy

$$\ell_n = \text{length}(\alpha_n) = \frac{1}{t_n}\text{length}(\gamma_n) \leq \frac{1}{t_n}d_X(x_n, y_n) + \delta\frac{\|x_n - y_n\|}{t_n} \leq (K+\delta)\frac{\|x_n - y_n\|}{t_n}$$

Since $\frac{\|x_n - y_n\|}{t_n} \to \|v - w\|$, we can see that $\{\ell_n\}_{n\in\mathbb{N}}$ is bounded. Passing to a subsequence, assume $\ell_n \leq \ell = (K+\delta)\|v - w\| + 1$. For each n, let $\tilde{\alpha}_n : [0, \ell_n] \to X$ be a reparametrisation by arc length of α_n. Also we let $\beta_n : [0, \ell] \to X$ be the curve given by

$$\beta_n(t) = \begin{cases} \tilde{\alpha}_n(t), & \text{if } t \in [0, \ell_n] \\ \tilde{\alpha}_n(\ell_n), & \text{if } t \in [\ell_n, \ell] \end{cases}$$

Each $\tilde{\alpha}_n$ is parametrised by arc length so for every $n \in \mathbb{N}$,

$$\|\beta_n(s) - \beta_n(t)\| \leq |s - t|, \quad \forall s, t \in [0, \ell]$$

The family $\{\beta_n\}$ is bounded and equicontinuous. By the Arzelà-Ascoli theorem, a subsequence $\{\beta_{n_k}\}$ converges uniformly to a continuous curve $\alpha : [0, \ell] \to T_0X$.
For sufficiently large k,

$$\text{length}(\beta_{n_k}) \leq (K+\delta)\|v - w\| + \delta$$

implying $\text{length}(\alpha) \leq (K+\delta)\|v - w\| + \delta$. Thus $d_{T_0X}(v, w) \leq (K+\delta)\|v - w\| + \delta$. Since δ was arbitrarily chosen, we see that $d_{T_0X}(v, w) \leq K\|v - w\|$.
Finally, we prove that for any pair of vectors $x, y \in T_0X$, their inner distance in the cone T_0X is at most $(1+K)\|x - y\|$. We denote by d_{T_0X} the inner distance of T_0X. If $x = sy$ for some $s \in \mathbb{R}$, then the line segment connecting x and y is contained in T_0X, so

$$d_{T_0X}(x, y) = \|x - y\| \leq (1+K)\|x - y\|$$

Otherwise, assume $x \neq sy$ for all $s \in \mathbb{R}$ and $x \neq 0$, $y \neq 0$. Define $y^* = ty$ with $t = \frac{\|x\|}{\|y\|}$. The geometric configuration ensures

$$\|x - y\| \geq \|x - y^*\|, \quad \|x - y\| \geq \|y^* - y\|$$

Thus,

$$\begin{aligned}(1+K)\|x - y\| &= \|x - y\| + K\|x - y\| \\ &\geq \|y^* - y\| + K\|x - y^*\| \\ &\geq d_{T_0X}(y, y^*) + d_{T_0X}(x, y^*) \\ &\geq d_{T_0X}(x, y)\end{aligned}$$

completing the proof. □

6.2 Reduced Tangent Cones

Definition 6.4. A subset $X \subset \mathbb{R}^n$ is called *subanalytic* at $x \in \mathbb{R}^n$ if there exists an open neighborhood U of x in $\mathbb{R}^n$ and a bounded semianalytic subset $S \subset \mathbb{R}^n \times \mathbb{R}^m$, for some m, such that

$$U \cap X = \pi(S),$$

where $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the orthogonal projection map.

Definition 6.5. A subset $X \subset \mathbb{R}^n$ is called *subanalytic in* $\mathbb{R}^n$ if X is subanalytic at each point of $\mathbb{R}^n$.

Let $X \subseteq \mathbb{R}^m$ be a subanalytic set containing the origin.

Definition 6.6. A point $x \in \partial X'$ is called *simple* if there exists an open $U \subseteq \mathbb{R}^{m+1}$ containing x such that
1. The connected components $X_1, \ldots, X_r$ of $(X' \cap U) \setminus \partial X'$ are topological manifolds of dimension equal to $\dim X$.
2. For each i, the union $(X_i \cup \partial X') \cap U$ forms a topological manifold with boundary.
The collection of all simple points in ∂X_0 is denoted $\mathrm{Simp}(\partial X,)$.

Definition 6.7. We consider the spherical blowing-up at the origin of $\mathbb{R}^m$,

$$\rho_m : \mathbb{S}^{m-1} \times [0, +\infty) \longrightarrow \mathbb{R}^m, \quad (x, r) \mapsto rx.$$

For $X \subseteq \mathbb{R}^m$ subanalytic with $0 \in X$, define $k_X : \mathrm{Simp}(\partial X_0) \to \mathbb{N}$ as

$$k_X(x) = \text{number of connected components of } \rho_m^{-1}(X \setminus \{0\}) \cap U,$$

where U is an adequately small neighborhood of x.

Definition 6.8. A subanalytic set $X \subseteq \mathbb{R}^m$ with $0 \in X$ is said to have a *reduced tangent cone* at 0 if $k_X(x) = 1$ for every $x \in \mathrm{Simp}(\partial X_0)$.

Theorem 6.9. *If $X \subseteq \mathbb{R}^m$ is subanalytic, contains 0, and is Lipschitz normally embedded at 0, then its tangent cone at 0 is reduced.*

Proof. Assume for contradiction that there exists $x = (x', 0) \in \mathrm{Simp}(\partial X') \subseteq S_0X \times \{0\}$ where $S_0X = T_0X \cap \mathbb{S}^{m-1}$ with $k_X(x) \geq 2$. There exist $\delta, \varepsilon > 0$ such that $k_X(y) = k_X(x) = k$ for all $y \in \partial X' \cap B_\varepsilon(x)$. The set $X' \cap U_{\delta,\varepsilon} \setminus \partial X'$ splits into k components, where

$$U_{\delta,\varepsilon} = \{(w, s) \in S^{m-1} \times [0, +\infty] \mid \|w - x'\| < \varepsilon, 0 \leq s \leq \delta\}$$

Let X_1, X_2 be two connected components of $X' \cap U_{\delta,\varepsilon} \setminus \partial X'$. Define the conical region

$$C_{\delta,\varepsilon} = \{v \in \mathbb{R}^m \setminus \{0\} \mid \|v - sx'\| < s\varepsilon,\ 0 < s < \delta\}$$

For each n, we can select $x_n \in \rho(X_1)$, $y_n \in \rho(X_2)$ with $\|x_n\| = \|y_n\| = t_n$ such that

$$\lim_{n\to\infty} \frac{x_n}{t_n} = \lim_{n\to\infty} \frac{y_n}{t_n} = x'$$

By taking a subsequence if necessary, we can assume $x_n, y_n \in C_{\delta,\varepsilon/2}$. If $\gamma_n : [0,1] \to X$ is a curve connecting x_n to y_n, there exists $t_0 \in [0,1]$ such that $\gamma_n(t_0) \notin C_{\delta,\varepsilon}$, since x_n and y_n belong to different connected components of $X \cap C_{\delta,\varepsilon}$. implying

$$\text{length}(\gamma_n) \geq \varepsilon t_n \quad \Rightarrow \quad d_X(x_n, y_n) \geq \varepsilon t_n$$

However, since X is Lipschitz normally embedded, there exists $K > 0$ such that

$$d_X(v, w) \leq K\|v - w\| \quad , \forall v, w \in X$$

Thus,

$$K \left\| \frac{x_n}{t_n} - \frac{y_n}{t_n} \right\| \geq \varepsilon, \quad \forall n \in \mathbb{N}$$

This leads to a contradiction, since

$$\lim_{n\to\infty} \frac{x_n}{t_n} = \lim_{n\to\infty} \frac{y_n}{t_n}$$

□

Corollary 6.10. *If $X \subseteq \mathbb{R}^m$ is subanalytic, Lipschitz normally embedded at 0, then T_0X is both reduced and Lipschitz normally embedded.*

7 Lipschitz Knot Theory

A link at the origin of an isolated singularity of a two-dimensional semialgebraic surface in $\mathbb{R}^4$ is a topological knot in S^3. We can thus study the overlap of knot theory and Lipschitz geometry. Specifically, we will state and prove the 'Universality Theorem' from [5]

7.1 Concepts

We analyse germs of two-dimensional semialgebraic sets at the origin in $\mathbb{R}^4$.

Definition 7.1. Two germs $(X, 0)$ and $(Y, 0)$ are *outer bi-Lipschitz equivalent* if there exists a bi-Lipschitz homeomorphism $H : (X, 0) \to (Y, 0)$ with respect to the Euclidean metric. These germs are *semialgebraic outer bi-Lipschitz equivalent* if the map H can be chosen to be semialgebraic.

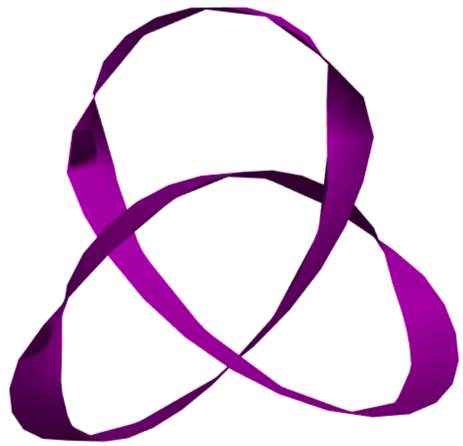
Definition 7.2. The germs are *ambient bi-Lipschitz equivalent* if there exists a bi-Lipschitz homeomorphism $H_e : (\mathbb{R}^4, 0) \to (\mathbb{R}^4, 0)$ preserving orientation and satisfying $H_e(X) = Y$. Furthermore, the germs are *semialgebraic ambient bi-Lipschitz equivalent* if H_e can be chosen to be semialgebraic.

A fundamental tool in relating local geometry to knot theory is the notion of a link of a singularity. Intuitively, when one intersects a germ with a small sphere centered at the singular point, the resulting set captures the topology of the singularity. In the next definition we formalise the concept of the origin link and, by extension, the tangent link of a germ.

Definition 7.3. The *origin link* L_X of a germ X is the ambient equivalence class of $X \cap S^3_{0,\varepsilon}$ for small $\varepsilon > 0$. The *tangent link* is the origin link of X's tangent cone.

To connect the study of surface germs with classical knot theory, one may represent a knot by a smooth semialgebraic circle embedded in S^3. However, to probe the structure of the singularity, we construct a characteristic band. This is an embedded surface that carries the knot along its boundary. By taking the cone over such a band or its boundary, we obtain a model of a surface germ that reflects the geometry of the original knot.

Definition 7.4. The knot K is realised as a smooth semialgebraic circle embedded in S^3. Let $F_K \subset S^3$ be a smooth semialgebraic embedded surface diffeomorphic to $S^1 \times [-1, 1]$ with boundary components K_e, K'_e of the boundary ∂F_K isotopic to a knot K and the linking number of the components K_e and K'_e is zero. F_K is a *characteristic band* of K. The cone over F_K denoted $Y_K = \text{Cone}(F_K)$, and the cone over ∂F_K denoted $X_K = \text{Cone}(\partial F_K)$, are called the *characteristic cones* for K.



Characteristic band around trefoil knot

Definition 7.5. Let $\rho \in S^1$ and $l \in [-1, 1]$. Then $(\rho, l) \in F_K$. For an interior point $\xi = (\rho_0, 0) \in F_K$, a *slice* S_K is

$$S_K = \{(\rho, l) \in F_K \mid |\rho - \rho_0| \leq \epsilon\}$$

7.2 Universality Theorem

Lemma 7.6. *Let $X, Y \subset \mathbb{R}^n$ be subanalytic sets. If X and Y are bi-Lipschitz homeomorphic, then there exists a bi-Lipschitz homeomorphism $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that*

$$\varphi(X \times \{0\}) = \{0\} \times Y$$

Proof. Let $\phi : X \to Y$ be a bi-Lipschitz homeomorphism. The McShane–Whitney–Kirszbraun theorem states there exist Lipschitz maps $\overline{\phi} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\overline{\phi}|_X = \phi$ and another Lipschitz map $\overline{\psi} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\overline{\psi}|_Y = \phi^{-1}$. Define $\varphi, \psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ as follows

$$\varphi(x, y) = (x - \overline{\psi}(y + \overline{\phi}(x)), y + \overline{\phi}(x))$$

and

$$\psi(z, w) = (z + \overline{\psi}(w), w - \overline{\phi}(z + \overline{\psi}(w)))$$

Since φ and ψ are compositions of Lipschitz maps, they are also Lipschitz maps. Next, we show that $\psi = \varphi^{-1}$. For any $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$\psi(\varphi(x, y)) = \psi(x - \overline{\psi}(y + \overline{\phi}(x)), y + \overline{\phi}(x))$$

$$= (x - \overline{\psi}(y + \overline{\phi}(x)) + \overline{\psi}(y + \overline{\phi}(x)), y + \overline{\phi}(x) - \overline{\phi}(x - \overline{\psi}(y + \overline{\phi}(x)) + \overline{\psi}(y + \overline{\phi}(x))))$$

$$= (x, y + \overline{\phi}(x) - \overline{\phi}(x))$$
$$= (x, y)$$

Similarly, for $(z, w) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$\varphi(\psi(z, w)) = \varphi(z + \overline{\psi}(w), w - \overline{\phi}(z + \overline{\psi}(w)))$$
$$= (z + \overline{\psi}(w) - \overline{\psi}(w - \overline{\phi}(z + \overline{\psi}(w))) + \overline{\phi}(z + \overline{\psi}(w)), w - \overline{\phi}(z + \overline{\psi}(w)) + \overline{\phi}(z + \overline{\psi}(w)))$$
$$= (z + \overline{\psi}(w) - \overline{\psi}(w), w)$$
$$= (z, w)$$

Thus, $\psi = \varphi^{-1}$. Finally, it is clear that $\varphi(X \times \{0\}) = \{0\} \times Y$. □

Theorem 7.7 (Sampaio's Theorem). *Let $X, Y \subset \mathbb{R}^m$ be subanalytic sets. If the germs (X, x_0) and (Y, y_0) are bi-Lipschitz homeomorphic, then $(T_{x_0}X, x_0)$ and $(T_{y_0}Y, y_0)$ are also bi-Lipschitz homeomorphic.*

Proof. We can assume that $x_0 = y_0 = 0$. By Lemma 7.6, we can suppose that there exists a bi-Lipschitz map $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ such that $\varphi(X) = Y$. Let $K > 0$ be a constant such that

$$\frac{1}{K}\|x - y\| \leq \|\varphi(x) - \varphi(y)\| \leq K\|x - y\|, \quad \forall x, y \in \mathbb{R}^m$$

Let $\psi = \varphi^{-1}$. We define two sequences of maps: for each $n \in \mathbb{N}$, we define the mappings $\varphi_n : \overline{B_1^m} \to \mathbb{R}^m$ and $\psi_n : \overline{B_K^m} \to \mathbb{R}^m$ by

$$\varphi_n(v) = n\varphi\left(\frac{v}{n}\right), \quad \psi_n(v) = n\psi\left(\frac{v}{n}\right)$$

We note that for any $n \in \mathbb{N}$, we have

$$\frac{1}{K}\|u - v\| \leq \|\varphi_n(u) - \varphi_n(v)\| \leq K\|u - v\|, \quad \forall u, v \in \overline{B_1^m}$$

and

$$\frac{1}{K}\|u - v\| \leq \|\psi_n(u) - \psi_n(v)\| \leq K\|u - v\|, \quad \forall u, v \in \overline{B_K^m}$$

Then, by the Arzelà–Ascoli theorem, there exists a subsequence $\{n_j\} \subset \mathbb{N}$ and mappings $d\varphi : \overline{B_1^m} \to \mathbb{R}^m$ and $d\psi : \overline{B_K^m} \to \mathbb{R}^m$ such that $\varphi_{n_j} \to d\varphi$ and $\psi_{n_j} \to d\psi$ uniformly as $j \to \infty$. Clearly,

$$\frac{1}{K}\|u - v\| \leq \|d\varphi(u) - d\varphi(v)\| \leq K\|u - v\|$$

and

$$\frac{1}{K}\|z - w\| \leq \|d\psi(z) - d\psi(w)\| \leq K\|z - w\|$$

Let $U = d\varphi(B_1^m)$. $d\varphi$ is a continuous and injective map so U is an open set.
Let $v \in T_0X \cap B_1^m$ and define $w = d\varphi(v) = \lim_{j\to\infty} \frac{\varphi(t_j v)}{t_j}$ with $t_j = \frac{1}{n_j}$.

$$
\begin{aligned}
\|d\psi(w) - v\| &= \left\| \lim_{j\to\infty} \frac{\psi(t_j w)}{t_j} - v \right\| \\
&= \lim_{j\to\infty} \left\| \frac{\psi(t_j w)}{t_j} - \frac{t_j v}{t_j} \right\| \\
&= \lim_{j\to\infty} \frac{1}{t_j} \|\psi(t_j w) - t_j v\| = \lim_{j\to\infty} \frac{1}{t_j} \|\varphi^{-1}(t_j w) - \varphi^{-1}(\varphi(t_j v))\| \\
&\le \lim_{j\to\infty} \frac{K}{t_j} \|t_j w - \varphi(t_j v)\| \le \lim_{j\to\infty} K \left\| w - \frac{\varphi(t_j v)}{t_j} \right\| \\
&= 0.
\end{aligned}
$$

Thus, $d\psi(w) = d\psi(d\varphi(v)) = v$ for all $v \in B_1^m$, i.e., $d\psi \circ d\varphi = \mathrm{id}_{B_1^m}$ and similarly, $d\varphi \circ d\psi|_U = \mathrm{id}_U$.

Let $v \in T_0X \cap B_1^m$, then there exists $\alpha : [0, \varepsilon) \to X$ such that $\alpha(t) = tv + o(t)$. Using the facts that $\varphi(\alpha(t)) \in Y$ for all $t \in [0, \varepsilon)$ and φ is Lipschitz, we have

$$
\varphi(\alpha(t)) = \varphi(tv) + o(t).
$$

By definition of $d\varphi$,

$$
\varphi(t_j v) = t_j d\varphi(v) + o(t_j),
$$

thus

$$
d\varphi(v) = \lim_{j\to\infty} \varphi_{n_j}(v) = \lim_{j\to\infty} \frac{\varphi(t_j v)}{t_j} = \lim_{j\to\infty} \frac{\varphi(\alpha(t_j))}{t_j} \in T_0Y.
$$

Therefore, $d\varphi(T_0X \cap B_1^m) \subset T_0Y$.

Similarly, we have $d\psi(T_0Y \cap U) \subset T_0X$.

Thus, $d\varphi : T_0X \cap B_1^m \to T_0Y \cap U$ is a bi-Lipschitz map.

□

Definition 7.8. Let $\beta > 1$ be a rational number. For a fixed $t \ge 0$, define $Z = \bigcup_{t\ge 0} Z_t$ where $Z_t = \{(x, y) \in \mathbb{R}^2 \mid |x| \le t, |y| \le t\}$ Let W_t^+ be the subset of Z_t bounded by the line segment $I_t^+ = \{(x, y) \mid |x| \le t, y = t\}$ and the union J_t^+ of the two line segments connecting the endpoints of I_t^+ with the point $(0, t^\beta)$. Define $W_t^- = \{(x, y) \mid (x, -y) \in W_t^+\}$ and similarly, $J_t^- = \{(x, y) \mid (x, -y) \in J_t^+\}$. We set $W_t = W_t^+ \cup W_t^-$ and let $W = \bigcup_{t\ge 0} W_t \subset \mathbb{R}^3$. We define a β-bridge as the surface germ $B_\beta = \bigcup_{t\ge 0}(J_t^+ \cup J_t^-)$. We see that the tangent cone of W is the set $\{|x| \le |y| \le z\}$ and the tangent cone of B_β is the surface germ $\{|x| = |y| \le z\}$.

Theorem 7.9 (Universality Theorem)**.** *Let $K \subset S^3$ be a knot. Then one can associate to K a semialgebraic one-bridge surface germ $(X_K, 0)$ in $\mathbb{R}^4$ so that the following holds:*

1. *The link at the origin of each germ X_K is a trivial knot;*
2. *All germs X_K are outer bi-Lipschitz equivalent;*
3. *Two germs X_{K_1} and X_{K_2} are ambient semialgebraic bi-Lipschitz equivalent only if the knots K_1 and K_2 are isotopic.*

Proof. Let $F_K \subset S^3$ be a characteristic band for the knot K, with characteristic cones $\overline{Y}_K = \text{Cone}(F_K)$ and $\overline{X}_K = \text{Cone}(\partial F_K)$. Let $S_K \subset F_K$ be a slice. Define the semi-algebraic bi-Lipschitz homeomorphism

$$\varphi_K : S_K \to Z_1 = \{(x, y) \in \mathbb{R}^2 \mid |x| \leq 1, |y| \leq 1\}, \quad (\rho, l) \mapsto \left(\frac{\rho - \rho_0}{\epsilon}, l\right)$$

Let $M_K = \{t\sigma \mid t \geq 0, \sigma \in S_K\}$ be the cone over S_K. We define the bi-Lipschitz homeomorphism

$$\Phi_K : M_K \to Z \subset \mathbb{R}^3, \quad \Phi_K(t\sigma) = t\varphi_K(\sigma), \quad \sigma \in S_K$$

Define $V_K = \Phi_K^{-1}(W)$ for $W \subset \mathbb{R}^3$, and construct

$$Y_K = (\overline{Y}_K \setminus M_K) \cup V_K, \quad X_K = \partial Y_K$$

Here, X_K is a one-bridge surface germ where a β-bridge B_β replaces part of the surface germ $\overline{X_K}$ inside M_K.
1. The link at the origin of X_K bounds the closure of $F_K \setminus S_K$, which is a homeomorphic to a disk and is therefore a trivial knot.
2. For knots K_1, K_2, define a semialgebraic bi-Lipschitz map

$$\Psi : \overline{Y}_{K_1} \to \overline{Y}_{K_2}, \quad (\rho, l, t) \in \overline{Y}_{K_1} \mapsto (\rho, l, t) \in \overline{Y}_{K_2}.$$

We have $\Psi(M_{K_1}) = M_{K_2}$. By definition of Φ_{K_1}, Φ_{K_2} we have that $\Psi(Y_{K_1}) = Y_{K_2}$ and that $\Psi(X_{K_1}) = X_{K_2}$
3. For any knot K, the link of the tangent cone T_0X_K of the set X_K is the union of two knots isotopic to K, with a single common point. Therefore, if the knots K_1 and K_2 are not isotopic, then the tangent cones $T_0X_{K_1}$ and $T_0X_{K_2}$ are not ambient topologically equivalent, a contradiction to Sampaio's theorem (Theorem 7.7). In our case, the links of the tangent cones are not even ambient topologically equivalent.
Thus, Universality Theorem holds. □

Remark 7.10. Even though each X_K has a topologically trivial link at the singular point, the ambient Lipschitz geometry of X_K encodes the full knot type of K. All X_K share the same outer conical geometry, but no two of them can be mapped onto each other by an ambient bi-Lipschitz homeomorphism unless the underlying knots are equivalent. This fact means that any knot invariant can, in principle, be detected from the Lipschitz geometry of the corresponding surface germ.

Example 7.11 (Trefoil Knot vs. Unknot)**.** To illustrate Universality Theorem, consider K to be the trefoil knot and let K' be the unknot. By the construction in the theorem, we obtain two one-bridge surface germs $(X_K, 0)$ and $(X_{K'}, 0)$ in $\mathbb{R}^4$. Both germs have a link at the origin which is an unknot by condition (1) above. In fact, X_K and $X_{K'}$ are indistinguishable from the outside, they are outer bi-Lipschitz equivalent by condition (2). However, since the trefoil K is not isotopic to the unknot K', condition (3) guarantees that there is no ambient bi-Lipschitz homeomorphism of $\mathbb{R}^4$ taking X_K to $X_{K'}$. Thus, the ambient Lipschitz geometry successfully distinguishes a trefoil from an unknot.

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